Quantum White Noise Derivatives and Transformations

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International Conference on

Quantum Probability and Related Topics

August 14–17, 2010

JNCASR, Bangalore

CONTENTS

Rigging of Boson Fock Space

We start with the Hilbert space $H = L^2(\mathbf{R}, dt)$ with norm $|\cdot|_0$.

Let $A = 1 + t^2 - \dfrac{d^2}{dt^2}$ be the harmonic oscillator. For each $p \geq 0$ we put

$$E_p \equiv \mathrm{Dom}\,(A^p) \subset H; \qquad E_{-p} = E_p^*.$$

$$\mathscr{S}(\mathbf{R}) \cong E \equiv \operatorname*{proj\,lim}_{p\to\infty} E_p \subset E_p \subset H \subset E_{-p} \subset \operatorname*{ind\,lim}_{p\to\infty} E_{-p} \equiv E^* \cong \mathscr{S}'(\mathbf{R}).$$

By taking Boson Fock spaces and their (projective and inductive) limit spaces, we have

$$(E) = \operatorname*{proj\,lim}_{p\to\infty} \Gamma(E_p) \subset \Gamma(H) = \bigoplus_{n=0}^{\infty} H^{\widehat{\otimes} n} \subset (E)^* = \operatorname*{ind\,lim}_{p\to\infty} \Gamma(E_{-p}),$$

where $\Gamma(H)$ is the *Boson Fock space* over H is defined by

$$\Gamma(H) = \left\{ \phi = (f_n)_{n=0}^{\infty}\,;\, f_n \in H^{\widehat{\otimes} n},\, \|\phi\|_0^2 = \sum_{n=0}^{\infty} n!\,|f_n|_0^2 < \infty \right\},$$

and so we have the nuclear Rigging of Boson Fock space

$$(E) \subset \Gamma(H) \subset (E)^*.$$

White Noise Operators

• **An element of $\mathscr{L}((E),(E)^*)$ is called a *white noise operator* which is a kind of generalized operator based on the Gelfand triple:**

$$(E) \subset \Gamma(H) \subset (E)^*.$$

Annihilation and creation operators

• **For each $x \in \mathscr{S}'(\mathbf{R})$, the *annihilation operator $a(x)$* is defined by**

$$a(x)\left(0,\cdots,0,\xi^{\otimes n},0,\cdots\right) = \left(0,\cdots,0,\langle x,\,\xi\rangle\,\xi^{\otimes(n-1)},0,\cdots\right).$$

Then for each $x \in \mathscr{S}'(\mathbf{R})$, $a(x) \in \mathscr{L}((E),(E))$. The adjoint $a^*(x) \in \mathscr{L}((E)^*,(E)^*)$ of $a(x)$ is called the *creation operator* and we have

$$a^*(x)\left(0,\cdots,0,\xi^{\otimes n},0,\cdots\right) = \left(0,\cdots,0,x\widehat{\otimes}\xi^{\otimes n},0,\cdots\right).$$

<u>`NOTE.`</u> **Let $\zeta \in \mathscr{S}(\mathbf{R})$. Then we have**

$$a(\zeta) \in \mathscr{L}((E)^*,(E)^*), \qquad a^*(\zeta) \in \mathscr{L}((E),(E)).$$

Quantum White Noise and Integral Kernel Operators

Put

$$a_t \equiv a(\delta_t), \qquad a_t^* \equiv a^*(\delta_t).$$

Then the pair $\{a_t, a_t^*; t \in \mathbf{R}\}$ is called the *quantum white noise*.

For each $\kappa_{l,m} \in (E^*)^{\otimes(l+m)}$, the *integral kernel operator* $\Xi_{l,m}(\kappa_{l,m})$ is formally expressed as

$$\Xi_{l,m}(\kappa_{l,m}) = \int_{\mathbf{R}^{l+m}} \kappa_{l,m}(s_1, \dots, s_l, t_1, \dots, t_m) a_{s_1}^* \cdots a_{s_l}^* a_{t_1} \cdots a_{t_m} ds_1 \cdots ds_l dt_1 \cdots dt_m.$$

For exponential vector,

$$\phi_\xi = \left(1, \xi, \frac{\xi^{\otimes 2}}{2!}, \cdots, \frac{\xi^{\otimes n}}{n!}, \cdots\right), \qquad \xi \in E,$$

we have

$$\langle\langle \Xi_{l,m}(\kappa_{l,m})\phi_\xi, \phi_\eta \rangle\rangle = \langle \kappa_{l,m}, \xi^{\otimes m} \otimes \eta^{\otimes l} \rangle e^{\langle \xi, \eta \rangle},$$

where $\langle\langle \cdot, \cdot \rangle\rangle$ is the canonical complex bilinear form on $(E)^* \times (E)$.

Generalized Gross Laplacians

• **For each** $U \in \mathscr{L}(E, E^*)$ **there exists a unique** $\tau_U \in E^* \times E^*$ **such that**

$$\langle \tau_U, \eta \otimes \xi \rangle = \langle U\xi, \eta \rangle, \qquad \xi, \eta \in E,$$

and then the *generalized Gross Laplacian* $\Delta_{\mathrm{G}}(U)$ **is defined by**

$$\Delta_{\mathrm{G}}(U) = \Xi_{0,2}(\tau_U) = \int_{\mathbf{R}^2} \tau_U(t_1, t_2) a_{t_1} a_{t_2} dt_1 dt_2 \in \mathscr{L}((E), (E)).$$

In particular, for $U = I$, $\Delta_{\mathrm{G}}(I)$ **is called the *Gross Laplacian*** Δ_{G}**.**

• **The adjoint of** $\Delta_{\mathrm{G}}(U)$ **is denoted by** $\Delta_{\mathrm{G}}^*(U)$ **which is represented by**

$$\Delta_{\mathrm{G}}^*(U) = \Xi_{2,0}(\tau_U) = \int_{\mathbf{R}^2} \tau_U(t_1, t_2) a_{t_1}^* a_{t_2}^* dt_1 dt_2 \in \mathscr{L}((E)^*, (E)^*).$$

Conservation Operators

• **For each** $S \in \mathscr{L}(E, E^*)$**, the operator** $\Lambda(S)$ **defined by**

$$\Lambda(S) = \Xi_{1,1}(\tau_S) = \int_{\mathbf{R}^2} \tau_S(s, t) a_s^* a_t ds dt \in \mathscr{L}((E), (E)^*).$$

is called the *conservation operator*. In particular, $N \equiv \Lambda(I)$ **is the *number operator*.**

Quantum White Noise Derivatives

Fock Expansion and Motivation

• **A fundamental consequence of quantum white noise theory that every white noise operator $\Xi \in \mathscr{L}((E),(E)^*)$ admits a *Fock expansion*:**

$$\Xi = \sum_{l,m=0}^{\infty} \Xi_{l,m}(\kappa_{l,m})$$

with

$$\Xi_{l,m}(\kappa_{l,m}) = \int_{\mathbf{R}^{l+m}} \kappa_{l,m}(s_1,\dots,s_l,t_1,\dots,t_m) a_{s_1}^* \cdots a_{s_l}^* a_{t_1} \cdots a_{t_m} ds_1 \cdots ds_l dt_1 \cdots dt_m.$$

In this sense, every white noise operator can be considered as a "function" of quantum white noise:

$$\Xi = \Xi(a_s, a_t^*; s,t \in \mathbf{R}),$$

and so we are naturally interested in its derivatives:

$$D_t^+ \Xi = \frac{\delta \Xi}{\delta a_t^*}, \qquad D_t^- \Xi = \frac{\delta \Xi}{\delta a_t}.$$

Creation- and Annihilation-Derivatives

For any white noise operator $\Xi \in \mathscr{L}((E),(E)^*)$ and $\zeta \in E$, the commutators

$$[a(\zeta),\Xi] = a(\zeta)\Xi - \Xi a(\zeta), \qquad -[a^*(\zeta),\Xi] = \Xi a^*(\zeta) - a^*(\zeta)\Xi,$$

are well defined white noise operators, i.e., belongs to $\mathscr{L}((E),(E)^*)$. We define

$$D_\zeta^+\Xi = [a(\zeta),\Xi], \qquad D_\zeta^-\Xi = -[a^*(\zeta),\Xi].$$

DEFINITION. $D_\zeta^+\Xi$ **and** $D_\zeta^-\Xi$ **are respectively called the** ***creation derivative*** **and** ***annihilation derivative*** **of Ξ, and both together the** ***quantum white noise derivatives*** **of Ξ.**

THEOREM. For each $\zeta \in E$, the quantum white noise derivatives $D_\zeta^\pm$ are continuous linear operators from $\mathscr{L}((E),(E)^*)$ into itself.

THEOREM. Let $\zeta \in E$ and let $\Xi \in \mathscr{L}((E),(E)^*)$ with Fock expansion $\Xi = \sum_{l,m=0}^{\infty} \Xi_{l,m}(\kappa_{l,m})$. Then,

$$D_\zeta^-\Xi = \sum_{l=0}^{\infty}\sum_{m=1}^{\infty} m\Xi_{l,m-1}(\kappa_{l,m}*\zeta), \qquad D_\zeta^+\Xi = \sum_{l=1}^{\infty}\sum_{m=0}^{\infty} l\Xi_{l-1,m}(\zeta*\kappa_{l,m}).$$

Examples. **By using the *canonical commutation relation* (CCR): for each $\zeta \in E$, we have**

$$
\begin{aligned}
D_\zeta^- \Delta_{\mathrm{G}}(S) &= a(S\zeta + S^*\zeta), & D_\zeta^+ \Delta_{\mathrm{G}}(S) &= 0,\\
D_\zeta^- \Delta_{\mathrm{G}}^*(S) &= 0, & D_\zeta^+ \Delta_{\mathrm{G}}^*(S) &= a^*(S\zeta + S^*\zeta),\\
D_\zeta^- \Lambda(S) &= a^*(S\zeta), & D_\zeta^+ \Lambda(S) &= a(S^*\zeta),\\
D_\zeta^- a(x) &= \langle x, \zeta \rangle, & D_\zeta^+ a(x) &= 0,\\
D_\zeta^- a^*(x) &= 0, & D_\zeta^+ a^*(x) &= \langle x, \zeta \rangle,
\end{aligned}
$$

In fact, we obtain that $D_\zeta^+ \Delta_{\mathrm{G}}(S) = \left[a(\zeta), \int_{\mathbf{R}^2} \tau_S(t_1,t_2) a_{t_1} a_{t_2} dt_1 dt_2 \right] = 0$ **and**

$$
\begin{aligned}
a_{t_1} a_{t_2} a^*(\zeta) &= a_{t_1} \left[a^*(\zeta) a_{t_2} + \zeta(t_2) \right] = \left[a^*(\zeta) a_{t_1} + \zeta(t_1) \right] a_{t_2} + \zeta(t_2) a_{t_1}\\
&= a^*(\zeta) a_{t_1} a_{t_2} + \zeta(t_1) a_{t_2} + \zeta(t_2) a_{t_1}
\end{aligned}
$$

and so

$$
\begin{aligned}
D_\zeta^- \Delta_{\mathrm{G}}(S) &= \Delta_{\mathrm{G}}(S) a^*(\zeta) - a^*(\zeta) \Delta_{\mathrm{G}}(S) = \int_{\mathbf{R}^2} \left[\tau_S(t_1,t_2) \left(\zeta(t_1) a_{t_2} + \zeta(t_2) a_{t_1} \right) \right] dt_1 dt_2\\
&= a(S\zeta) + a(S^*\zeta).
\end{aligned}
$$

Wick Derivations

Wick Product

For any $\Xi_1, \Xi_2 \in \mathscr{L}((E),(E)^*)$, the *Wick product* (or *normal-ordered product*) $\Xi_1 \diamond \Xi_2 \in \mathscr{L}((E),(E$ is well–defined as satisfying the characteristic properties:

$$a_t \diamond \Xi = \Xi \diamond a_t = \Xi a_t\,, \qquad a_t^* \diamond \Xi = \Xi \diamond a_t^* = a_t^* \Xi.$$

- $(\mathscr{L}((E),(E)^*), \diamond)$ **is a commutative algebra.**

Wick Derivations

A continuous linear map $\mathscr{D} : \mathscr{L}((E)^*,(E)) \to \mathscr{L}((E),(E)^*)$ is called a *Wick derivation* if

$$\mathscr{D}(\Xi_1 \diamond \Xi_2) = (\mathscr{D}\Xi_1) \diamond \Xi_2 + \Xi_1 \diamond (\mathscr{D}\Xi_2), \qquad \Xi_1, \Xi_2 \in \mathscr{L}((E),(E)^*).$$

THEOREM. For each $\zeta \in E$, the creation and annihilation derivatives $D_\zeta^\pm$ are Wick derivations.

THEOREM. **Let** $\mathscr{D} : \mathscr{L}((E)^*, (E)) \to \mathscr{L}((E), (E)^*)$ **be a Wick derivation. Then there exist** $F, G \in E^* \otimes \mathscr{L}((E), (E)^*)$ **such that**

$$\mathscr{D} = \int_T F(t) \diamond D_t^+ dt + \int_T G(t) \diamond D_t^- dt.$$

For each $\Xi \in \mathscr{L}((E), (E)^*)$ **, the map**

$$D_{(\cdot)}^{\pm}\Xi : E \ni z \longmapsto D_z^{\pm}\Xi \in \mathscr{L}((E), (E)^*)$$

is continuous, i.e., $D_{(\cdot)}^{\pm}\Xi \in \mathscr{L}(E, \mathscr{L}((E), (E)^*)) \cong E^* \otimes \mathscr{L}((E), (E)^*)$**. Therefore, by the previous Theorem, we have**

$$D_{\Xi^{\pm}}^{\pm} \equiv \int_{\mathbf{R}_+} \left(D_t^{\pm}\Xi\right) \diamond D_t^{\pm} dt$$

is a Wick derivation from $\mathscr{L}((E)^*, (E))$ **into** $\mathscr{L}((E), (E)^*)$**.**

PROPOSITION. **Let** $\mathscr{T} = \sum_{m=0}^{\infty} \Xi_{1,m}(\kappa_{1,m}) \in \mathscr{L}((E), (E))$ **and** $\mathscr{U} = \sum_{l=0}^{\infty} \Xi_{l,1}(\kappa_{l,1}) \in \mathscr{L}((E)^*, (E)^*)$**. Then we have**

$$D_{\mathscr{T}^+}^{-}\Xi \equiv \Xi\mathscr{T} - \Xi \diamond \mathscr{T}, \qquad D_{\mathscr{U}^-}^{+}\Xi = \mathscr{U}\Xi - \Xi \diamond \mathscr{U}, \qquad \Xi \in \mathscr{L}((E), (E)^*).$$

Examples. **Let** $S \in \mathscr{L}(E, E^*)$**. If** $S \in \mathscr{L}(E, E) \cong E \otimes E^*$**, then** $\Lambda(S) = \Xi_{1,1}(\tau_S) \in \mathscr{L}((E), (E))$ **and so we obtain that**

$$D^-_{\Lambda(S)^+}(\Xi) = \Xi\Lambda(S) - \Xi \diamond \Lambda(S), \qquad \Xi \in \mathscr{L}((E), (E)^*).$$

Similarly, if $S \in \mathscr{L}(E^*, E^*) \cong E^* \otimes E$**, then** $\Lambda(S) = \Xi_{1,1}(\tau_S) \in \mathscr{L}((E)^*, (E)^*)$ **and so we obtain that**

$$D^+_{\Lambda(S)^-} = \Lambda(S)\Xi - \Xi \diamond \Lambda(S), \qquad \Xi \in \mathscr{L}((E), (E)^*),$$

and then we have

$$\begin{aligned}
D^-_{\Lambda(S_1)^+}\Lambda(K) &= \Lambda(KS_1), & D^+_{\Lambda(S_2)^-}\Lambda(K) &= \Lambda(S_2K), \\
D^-_{\Lambda(S_1)^+}\Delta^*_{\mathrm{G}}(A) &= 0, & D^+_{\Lambda(S_2)^-}\Delta^*_{\mathrm{G}}(A) &= \Delta^*_{\mathrm{G}}(S_2A + AS^*_2), \\
D^-_{\Lambda(S_1)^+}\Delta_{\mathrm{G}}(B) &= \Delta_{\mathrm{G}}(S^*_1B + BS_1), & D^+_{\Lambda(S_2)^-}\Delta_{\mathrm{G}}(B) &= 0, \\
D^-_{\Lambda(S_1)^+}a^*(\zeta) &= 0, & D^+_{\Lambda(S_2)^-}a^*(\zeta) &= a^*(S_2\zeta), \\
D^-_{\Lambda(S_1)^+}a(\eta) &= a(S^*_1\eta), & D^+_{\Lambda(S_2)^-}a(\eta) &= 0.
\end{aligned}$$

Differential Equations Associated with Wick Derivations

Consider the following differential equation:

$$\mathscr{D}\Xi = G \diamond \Xi \qquad (\heartsuit)$$

associated with the Wick derivation $\mathscr{D} : \mathscr{L}((E),(E)^*) \to \mathscr{L}((E),(E)^*)$**.**

Wick Exponential: **For** $Y \in \mathscr{L}((E),(E)^*)$ **we define**

$$\mathrm{wexp}\, Y = \sum_{n=0}^{\infty} \frac{1}{n!} Y^{\diamond n},$$

whenever the series converges in $\mathscr{L}((E),(E)^*)$**.**

Differential Equations

THEOREM. **Assume that there exists an operator** $Y \in \mathscr{L}((E),(E)^*)$ **such that** $\mathscr{D}Y = G$ **and** $\mathrm{wexp}\, Y$ **is defined in** $\mathscr{L}((E),(E)^*)$**. Then every solution to** $(\heartsuit)$ **is of the form:**

$$\Xi = (\mathrm{wexp}\, Y) \diamond F,$$

where $F \in \mathscr{L}((E),(E)^*)$ **satisfying** $\mathscr{D}F = 0$**.**

Example. **Let us consider the (system of) differential equations:**

$$(\clubsuit) \qquad \begin{cases} D_\zeta^+ \Xi = 0, \\ D_\zeta^- \Xi = 0, \quad \zeta \in E. \end{cases}$$

If Ξ is a solution of the given system, Ξ is given by

$$\Xi = \sum_{m=0}^{\infty} \Xi_{0,m}(\kappa_{0,m}), \qquad \Xi = \sum_{l=0}^{\infty} \Xi_{l,0}(\kappa_{l,0}).$$

Consequently, a white noise operator Ξ satisfying $(\clubsuit)$ is a scalar operator. Thus, the irreducibility of the canonical commutation relation is reproduced.

Example. **Let us consider the differential equation:**

$$D_\zeta^- \Xi = 2a(\zeta) \diamond \Xi, \qquad \zeta \in E.$$

We need to find $Y \in \mathscr{L}((E),(E)^*)$ satisfying $D_\zeta^- Y = 2a(\zeta)$. In fact, $Y = \Delta_G$ is a solution. Moreover, it is easily verified that $\mathrm{wexp}\,\Delta_{\mathrm{G}}$ is defined in $\mathscr{L}((E),(E))$. Then, a general solution to the given equation is of the form:

$$\Xi = (\mathrm{wexp}\,\Delta_{\mathrm{G}}) \diamond F,$$

where $D_\zeta^- F = 0$ for all $\zeta \in E$.

Implementations Problems

• Canonical Commutation Relations

We now focus on the white noise operators of the form:

$$b(\zeta) = b_{S,T}(\zeta) = a(S\zeta) + a^*(T\zeta), \tag{0.1}$$

where $S, T \in \mathscr{L}(E, E)$ **and** $\zeta \in E$**. We note that** $b(\zeta) \in \mathscr{L}((E), (E)) \cap \mathscr{L}((E)^*, (E)^*)$**. The adjoint operators are given by**

$$b^*(\zeta) = b^*_{S,T}(\zeta) = a(T\zeta) + a^*(S\zeta). \tag{0.2}$$

THEOREM We maintain the notations and assumptions as above. The necessary and sufficient condition for $b(\zeta)$ **and** $b^*(\eta)$ **satisfying the CCR, i.e.,**

$$[b(\zeta), b(\eta)] = [b^*(\zeta), b^*(\eta)] = 0, \qquad [b(\zeta), b^*(\eta)] = \langle \zeta, \eta \rangle, \quad \zeta, \eta \in E,$$

is that

$$T^*S - S^*T = 0, \qquad S^*S - T^*T = I \qquad (\heartsuit 1).$$

Implementation Problem:

Our implementation problem is to find a white noise operator $U \in \mathscr{L}((E),(E)^*)$ satisfying

$$Ua(\zeta) = b(\zeta)U, \qquad (\heartsuit_1)$$
$$Ua^*(\zeta) = b^*(\zeta)U, \qquad (\heartsuit_2)$$

i.e.,

$$\begin{tikzcd} (E) \arrow[r, "U"] \arrow[d, "a(\zeta)"'] & (E)^* \arrow[d, "b(\zeta)"] \\ (E) \arrow[r, "U"'] & (E)^* \end{tikzcd} \qquad \begin{tikzcd} (E) \arrow[r, "U"] \arrow[d, "a^*(\zeta)"'] & (E)^* \arrow[d, "b^*(\zeta)"] \\ (E) \arrow[r, "U"'] & (E)^* \end{tikzcd}$$

which is equivalent to

$$D^+_{S\zeta}U = [a(\zeta - S\zeta) - a^*(T\zeta)] \diamond U, \qquad (\heartsuit_3)$$
$$(D^-_{\zeta} - D^+_{T\zeta})U = [a^*(S\zeta - \zeta) + a(T\zeta)] \diamond U. \qquad (\heartsuit_4)$$

<u>THEOREM</u> **Assume that S is invertible and that $T^*S = S^*T$. Then a white noise operator $U \in \mathscr{L}((E),(E)^*)$ satisfies the intertwining property:**

$$Ua(\zeta) = b(\zeta)U, \qquad \zeta \in E,$$

if and only if U is of the form

$$U = e^{-\frac{1}{2}\Delta^*_{\mathrm{G}}(TS^{-1})} \diamond \Gamma((S^{-1})^*) \diamond F,$$

where $F \in \mathscr{L}((E),(E)^*)$ is an arbitrary white noise operator satisfying $D^+_\zeta F = 0$ for all $\zeta \in E$.

<u>THEOREM</u> **Assume the following conditions:**

(i) S **is invertible;**

(ii) $T^*S = S^*T$;

(iii) $S^*S - T^*T = I$;

(iv) $ST^* = TS^*$.

Then a white noise operator $U \in \mathscr{L}((E),(E)^*)$ **satisfies the intertwining property:**

$$Ua^*(\zeta) = b^*(\zeta)U, \qquad \zeta \in E,$$

if and only if U **is of the form:**

$$U = e^{-\frac{1}{2}\Delta_{\mathrm{G}}^*(TS^{-1})}\Gamma((S^{-1})^*)\, e^{\frac{1}{2}\Delta_{\mathrm{G}}(S^{-1}T)} \diamond G,$$

where $G \in \mathscr{L}((E),(E)^*)$ **is an arbitrary white noise operator satisfying** $(D_\zeta^- - D_{T\zeta}^+)G = 0$ **for all** $\zeta \in E$.

<u>THEOREM</u> **Assumptions being the same as in the previous theorems, a white noise operator $U \in \mathscr{L}((E),(E)^*)$ satisfies the following intertwining properties:**

$$Ua(\zeta) = b(\zeta)U, \qquad Ua^*(\zeta) = b^*(\zeta)U, \qquad \zeta \in E,$$

if and only if U is of the form:

$$U = C\,e^{-\frac{1}{2}\Delta_{\mathrm{G}}^*(TS^{-1})}\Gamma((S^{-1})^*)\,e^{\frac{1}{2}\Delta_{\mathrm{G}}(S^{-1}T)},$$

where $C \in \mathbb{C}$.

The operator

$$U = C\,e^{-\frac{1}{2}\Delta_{\mathrm{G}}^*(TS^{-1})}\Gamma((S^{-1})^*)\,e^{\frac{1}{2}\Delta_{\mathrm{G}}(S^{-1}T)}$$

is called a Bogoliubov transformation.

General Implementation Problem:

For each given $\zeta_1, \eta_2 \in E$, $\eta_1, \zeta_2 \in E^*$, $S_1 \in \mathscr{L}(E,E)$, $S_2 \in \mathscr{L}(E^*,E^*)$ and $\mathscr{K} \in \mathscr{L}((E),(E))$, we verify certain conditions under which there exists a white noise operator $\mathscr{V} \in \mathscr{L}((E),(E)^*)$ such that

$$\mathscr{V}\left(a^*(\zeta_1)+a(\eta_1)+\Lambda(S_1)+\mathscr{K}\right) = \left(a^*(\zeta_2)+a(\eta_2)+\Lambda(S_2)\right)\mathscr{V}. \qquad (\spadesuit)$$

NOTE. **If $\mathscr{V}\mathscr{V}^\dagger = 1$, then we have**

$$\left\langle\left\langle \mathscr{V}^\dagger\phi_0, \overline{\mathscr{V}^\dagger\phi_0} \right\rangle\right\rangle = 1,$$

and so $\mathscr{V}^\dagger\phi_0$ gives a vector state. Therefore, the distribution of

$$a^*(\zeta_1)+a(\eta_1)+\Lambda(S_1)+\mathscr{K}$$

with respect to the vector state $\mathscr{V}^\dagger\phi_0$ coincides with the distribution of

$$a^*(\zeta_2)+a(\eta_2)+\Lambda(S_2)$$

with respect to the vacuum state ϕ_0.

Consider the modified implementation problem:

$$\mathscr{V}\left(a^*(\zeta_1)+a(\eta_1)+\Lambda(S_1)\right)+\mathscr{V}\diamond\widetilde{\mathscr{K}}=\left(a^*(\zeta_2)+a(\eta_2)+\Lambda(S_2)\right)\mathscr{V}, \qquad (\Diamond)$$

which is equivalent to

$$\mathscr{D}\mathscr{V}=\left(a^*(\zeta_2-\zeta_1)+a(\eta_2-\eta_1)+\Lambda(S_2-S_1)-\widetilde{\mathscr{K}}\right)\diamond\mathscr{V}$$

with the Wick derivation $\mathscr{V}$ **given by**

$$\mathscr{D}=D^-_{\zeta_1}+D^-_{\Lambda(S_1)^+}-\left(D^+_{\Lambda(S_2)^-}+D^+_{\eta_2}\right).$$

Put

$$\widetilde{\mathscr{K}}=\Delta^*_{\mathrm{G}}(A_3)+a^*(\zeta_3)+\Lambda(S_3)+a(\eta_3)+\Delta_{\mathrm{G}}(B_3)+\widetilde{k}, \qquad (\widetilde{\mathscr{K}})$$

where $A_3,B_3,S_3\in\mathscr{L}(E,E^*)$**,** $\zeta_3,\eta_3\in E^*$ **and** $\widetilde{k}\in\mathbf{C}$**,**

<u>THEOREM</u> If there exist $\zeta, \eta \in E^*$, $A \in \mathscr{A}_{\mathrm{s}}(E,E)$, $B \in \mathscr{A}_{\mathrm{s}}(E,E^*)$ and $S \in \mathscr{L}(E,E^*)$ such that the equations:

$$\begin{aligned} S_2A + AS_2^* &= A_3, \\ S_1^*B + BS_1 &= -B_3, \\ S\zeta_1 - S_2\zeta - 2A\eta_2 &= \zeta_2 - \zeta_3, \\ 2B\zeta_1 + S_1^*\eta - S^*\eta_2 &= -\eta_1 - \eta_3, \\ SS_1 - S_2S &= -S_3, \\ \langle \eta_2, \zeta\rangle - \langle \zeta_1, \eta\rangle &= \widetilde{k} \qquad (\clubsuit). \end{aligned}$$

are satisfied, then every solution $\mathscr{V}$ of $(\diamondsuit)$ is of the form

$$\mathscr{V} = e^{\Delta_{\mathrm{G}}^*(A)} e^{a^*(\zeta)} \Gamma(S) e^{a(\eta)} e^{\Delta_{\mathrm{G}}(B)} \diamond F$$

for a white noise operator $F \in \mathscr{L}((E),(E)^*)$ such that $\mathscr{D}F = 0$.

<u>THEOREM</u> If there exist $\zeta, \eta \in E^*$, $A \in \mathscr{A}_{\mathrm{s}}(E,E)$, $B \in \mathscr{A}_{\mathrm{s}}(E,E^*)$ and $S \in \mathscr{L}(E,E)$ such that $A_3 B \in \mathscr{L}_1(E,E)$, S is invertible and ($\clubsuit$) holds, then for any constant $C \in \mathbf{C} \setminus \{0\}$ the white noise operator of the form

$$\mathscr{V} = Ce^{\Delta_{\mathrm{G}}^*(A)} e^{a^*(\zeta)} \Gamma(S) e^{a(\eta)} e^{\Delta_{\mathrm{G}}(B)}$$

is a solution of ($\spadesuit$) with $\mathscr{K}$ given by

$$\mathscr{K} = \mathscr{V}^{-1} \left(\mathscr{V} \diamond \widetilde{\mathscr{K}} \right),$$

where $\widetilde{\mathscr{K}}$ is given as in ($\widetilde{\mathscr{K}}$).

COROLLARY. **If there exist $\zeta, \eta \in E^*$ and $S \in \mathscr{L}(E, E^*)$ such that the equations:**

$$S\zeta_1 - S_2\zeta = \zeta_2,$$
$$S_1^*\eta - S^*\eta_2 = -\eta_1,$$
$$SS_1 - S_2S = 0,$$
$$\langle \eta_2, \zeta \rangle - \langle \zeta_1, \eta \rangle = \widetilde{k}.$$

are satisfied with $\widetilde{k} = k$, then every solution $\mathscr{V}$ of

$$\mathscr{V}\,(a^*(\zeta_1) + a(\eta_1) + \Lambda(S_1) + k) = (a^*(\zeta_2) + a(\eta_2) + \Lambda(S_2))\,\mathscr{V}$$

is of the form

$$\mathscr{V} = e^{a^*(\zeta)}\Gamma(S)e^{a(\eta)} \diamond F$$

for a white noise operator $F \in \mathscr{L}((E), (E)^*)$ such that $\mathscr{D}F = 0$.

General Transformations

For each $U \in \mathscr{L}(E,E^*)$**,** $V \in \mathscr{L}(E,E)$**, we have**

$$\mathscr{G}_{U,V;\omega} = \Gamma(V)e^{\Delta_{\mathrm{G}}(U)}e^{a(\omega)}.$$

Then $\mathscr{G}_{U,V;\omega} \in \mathscr{L}((E),(E))$ **and the adjoint of** $\mathscr{G}_{U,V;\omega}$ **is denoted by** $\mathscr{F}_{U,V;\omega}$ **and then we have**

$$\mathscr{F}_{U,V;\omega} = e^{a^*(\omega)}e^{\Delta_{\mathrm{G}}^*(U)}\Gamma(V^*) \in \mathscr{L}((E)^*,(E)^*).$$

- **Υ–Transforms**

For each $U_1, U_2 \in \mathscr{L}(E,E^*)$**,** $V_1, V_2 \in \mathscr{L}(E,E)$ **and** $\omega_1, \omega_2 \in E^*$**, put**

$$\Upsilon_{U_2,V_2,\omega_2;U_1,V_1,\omega_1} = \mathscr{F}_{U_2,V_2;\omega_2} \underbrace{\mathscr{G}_{U_1,V_1:\omega_1}}_{Affine Transform} = \underbrace{e^{a^*(\omega_2)} \overbrace{\underbrace{e^{\Delta_{\mathrm{G}}^*(U_2)}\Gamma(V_2^*)}_{Fourier-Mehler} \; \underbrace{\Gamma(V_1)e^{\Delta_{\mathrm{G}}(U_1)}}_{Fourier-Gauss}}^{Bogoliubov} e^{a(\omega_1)}}_{Quantum\ Girsanov}$$

which is called the ***Υ–Transform*** **and motivated by**

$$e^{(a^*+a+c)^2} \quad \longleftrightarrow \quad e^{a^*+a^{*2}+a^*a+a^2+a+c} \quad \longleftrightarrow \quad ce^{a^*}e^{a^{*2}}e^{a^*a}e^{a^2}e^{a},$$

where c **is a constant.**

• Complex Gaussian Space

Let μ' be the Gaussian measure on $E^*_{\mathbf{R}}$ with mean 0 and variance $1/2$ of which the characteristic function is given by

$$\int_{E^*_{\mathbf{R}}} e^{i\langle x,\xi\rangle}\mu'(dx) = e^{-|\xi|_0^2/4}, \qquad \xi \in E_{\mathbf{R}}.$$

In view of the topological isomorphism $E^* \cong E^*_{\mathbf{R}} \times E^*_{\mathbf{R}}$, we define a probability measure $\nu = \mu' \times \mu'$ on E^* by

$$\nu(dz) = \mu'(dx)\mu'(dy), \qquad z = x+iy \in E^*.$$

The probability space (E^*,ν) is called the *complex Gaussian space*.

For each $\Xi \in \mathscr{L}((E),(E))$, with help of the resolution of the identity we have

$$\langle\langle \Xi\phi_\xi, \phi_\eta\rangle\rangle = \int_{E^*} \langle\langle \Xi\phi_\xi, \phi_z\rangle\rangle \langle\langle \phi_{\bar{z}}, \phi_\eta\rangle\rangle\, \nu(dz)$$

and so

$$\langle\langle \Xi_1\Xi_2\phi_\xi, \phi_\eta\rangle\rangle = \langle\langle \Xi_2\phi_\xi, \Xi_1^*\phi_\eta\rangle\rangle = \int_{E^*} \langle\langle \Xi_2\phi_\xi, \phi_z\rangle\rangle \langle\langle \phi_{\bar{z}}, \Xi_1^*\phi_\eta\rangle\rangle\, \nu(dz)$$

which is useful for the study of normal forms of operators.

- **Normal Ordered Forms**

THEOREM **Let** $A \in \mathscr{L}(E,E^*)$ **and** $B \in \mathscr{L}(E^*,E)$ **be symmetric such that for complete orthonormal basis** $\{e_k\}_{k=1}^{\infty} \subset E$ **of** H**,** $Ae_k = \alpha_k e_k$ **and** $Be_k = \beta_k e_k$ **with** $\alpha_k + \beta_k < 1$ **for** $k = 1,2,\cdots$**. Then we have**

$$e^{\Delta_{\mathrm{G}}(A)} e^{\Delta_{\mathrm{G}}^*(B)} = [\det(1-4BA)]^{-1/2} e^{\Delta_{\mathrm{G}}^*(B(1-4BA)^{-1})} \Gamma((1-4BA)^{-1}) e^{\Delta_{\mathrm{G}}(A(1-4BA)^{-1})}.$$

- **Unitary Implementations**

THEOREM **Let** $U_i \in \mathscr{L}_2(E,E)$ **and** $\omega_i \in E$**,** $i = 1,2$**. Let** $K \in \mathbf{C}$ **and** $B \in \mathscr{L}(E,E)$**. Then**

$$\Xi \equiv K e^{\Delta_{\mathrm{G}}^*(U_2)} \Gamma(B) e^{\Delta_{\mathrm{G}}(U_1)}$$

is unitary on $\Gamma(H)$ **if and only if** $\left[\det(1-4U_1^\dagger U_1)\right]^{1/4} = K = \left[\det(1-4U_2^\dagger U_2)\right]^{1/4}$ **and**

$$\overline{U_1} + B^\dagger \left[U_2 W_2\right]^* \overline{B} = 0 = \overline{U_2} + \overline{B}\left[U_1 W_1\right]^* B^\dagger;$$

$$U_1 + B^* \left[U_2^\dagger W_2\right]^* B = 0 = U_2 + B\left[U_1^\dagger W_1\right]^* B^*;$$

$$B^* W_2^* \overline{B} = I = B W_1^* B^\dagger, \qquad W_i = (1-4U_i^\dagger U_i)^{-1}, \quad i = 1,2.$$

THEOREM Let $U_i \in \mathscr{L}_2(E,E)$ **and** $\omega_i \in E$**,** $i = 1,2$**. Let** $K \in \mathbf{C}$ **and** $B \in \mathscr{L}(E,E)$**. Then**

$$\Xi \equiv K e^{a^*(\omega_2)} e^{\Delta_{\mathrm{G}}^*(U_2)} \Gamma(B) e^{\Delta_{\mathrm{G}}(U_1)} e^{a(\omega_1)}$$

is unitary on $\Gamma(H)$ **if and only if**

$$\left[\det(1-4U_1^\dagger U_1)\right]^{1/4} = K = \left[\det(1-4U_2^\dagger U_2)\right]^{1/4};$$

$$\overline{U_1} + B^\dagger \left[U_2 W_2\right]^* \overline{B} = 0 = \overline{U_2} + \overline{B}\left[U_1 W_1\right]^* B^\dagger;$$

$$U_1 + B^* \left[U_2^\dagger W_2\right]^* B = 0 = U_2 + B\left[U_1^\dagger W_1\right]^* B^*;$$

$$B^* W_2^* \overline{B} = I = B W_1^* B^\dagger;$$

$$B^*\left[\left(U_2^\dagger W_2 + (U_2^\dagger W_2)^*\right)\omega_2 + W_2^* \overline{\omega_2}\right] + \omega_1 = 0 = B\left[\left(U_1^\dagger W_1 + (U_1^\dagger W_1)^*\right)\omega_1 + W_1^* \overline{\omega_1}\right] + \omega_2;$$

$$2\Re\left\langle (U_2^\dagger W_2)^* \omega_2,\, \omega_2 \right\rangle + \langle (W_2^* \overline{\omega_2},\, \omega_2\rangle = 0 = 2\Re\left\langle (U_1^\dagger W_1)^* \omega_1,\, \omega_1 \right\rangle + \langle (W_1^* \overline{\omega_1},\, \omega_1\rangle\,,$$

where $W_i = (1-4U_i^\dagger U_i)^{-1}$**,** $i = 1,2$**.**

COROLLARY **Let** $U_i \in \mathscr{L}_2(E,E)$ **and** $\omega_i \in E$**,** $i = 1,2$**. Let** $K \in \mathbf{C}$ **and** $B \in \mathscr{L}(E,E)$**. Suppose that** U_1 **and** U_2 **are symmetric. Then**

$$\Xi \equiv K e^{a^*(\omega_2)} e^{\Delta_{\mathrm{G}}^*(U_2)} \Gamma(B) e^{\Delta_{\mathrm{G}}(U_1)} e^{a(\omega_1)}$$

is unitary on $\Gamma(H)$ **if and only if**

$$\left[\det(1 - 4U_1^\dagger U_1)\right]^{1/4} = K = \left[\det(1 - 4U_2^\dagger U_2)\right]^{1/4};$$
$$\overline{U_1} + B^\dagger \left[U_2 W_2\right]^* \overline{B} = 0 = \overline{U_2} + \overline{B}\left[U_1 W_1\right]^* B^\dagger;$$
$$B^* W_2^* \overline{B} = I = B W_1^* B^\dagger;$$
$$B^* \left[\left(U_2^\dagger W_2 + (U_2^\dagger W_2)^*\right)\omega_2 + W_2^* \overline{\omega_2}\right] + \omega_1 = 0 = B\left[\left(U_1^\dagger W_1 + (U_1^\dagger W_1)^*\right)\omega_1 + W_1^* \overline{\omega_1}\right] + \omega_2;$$
$$2\Re\left\langle (U_2^\dagger W_2)^* \omega_2,\, \omega_2 \right\rangle + \langle (W_2^* \overline{\omega_2},\, \omega_2\rangle = 0 = 2\Re\left\langle (U_1^\dagger W_1)^* \omega_1,\, \omega_1 \right\rangle + \langle (W_1^* \overline{\omega_1},\, \omega_1\rangle\,.$$

THEOREM **Let** $S \in \mathfrak{A}GL(E)$ **and** $T \in \mathscr{L}_2(E,E)$ **satisfying** $(\heartsuit 1)$ **and** $ST^* - TS^* = 0$**. Let** $\omega \in E$**. For the operator**

$$\mathscr{U}_{S,T,\omega} = e^{-a^*\left((S^{-1})^*\omega\right)} e^{-\Delta^*_{\mathrm{G}}(TS^{-1})} \Gamma(S^{-1}) e^{\Delta_{\mathrm{G}}(S^{-1}T)} e^{a\left((I-S^{-1}T(S^{-1})^*)\omega\right)},$$

$K\mathscr{U}_{S,T,\omega}$ **is unitary if and only if** $S^{-1}T, TS^{-1} \in \mathfrak{A}_2(E,E)$ **and** S,T **satisfy the following equations:**

$$\begin{gathered}
\left[\det(1-\left(S^{-1}T\right)^{\dagger} S^{-1}T)\right]^{1/4} = K = \left[\det(1-\left(TS^{-1}\right)^{\dagger} TS^{-1})\right]^{1/4}; \\
S^{-1}TS = \overline{STS^{-1}}; \\
S^{-1}T\left(S^{-1}\right)^{\dagger} = \overline{(S^{-1})^{\dagger} TS^{-1}}; \\
1-\left(TS^{-1}\right)^{\dagger} TS^{-1} = \overline{S^{-1}}\left(S^{-1}\right)^{*}; \\
1-\left(S^{-1}T\right)^{\dagger} S^{-1}T = \left(S^{-1}\right)^{*}\overline{S^{-1}}; \\
B^*\left[\left(U_2^{\dagger}W_2 + (U_2^{\dagger}W_2)^*\right)\omega_2 + W_2^*\overline{\omega_2}\right] + \omega_1 = 0 = B\left[\left(U_1^{\dagger}W_1 + (U_1^{\dagger}W_1)^*\right)\omega_1 + W_1^*\overline{\omega_1}\right] + \omega_2; \\
2\Re\left\langle (U_2^{\dagger}W_2)^*\omega_2,\, \omega_2\right\rangle + \langle (W_2^*\overline{\omega_2},\, \omega_2\rangle = 0 = 2\Re\left\langle (U_1^{\dagger}W_1)^*\omega_1,\, \omega_1\right\rangle + \langle (W_1^*\overline{\omega_1},\, \omega_1\rangle,
\end{gathered}$$

where $\omega_1 = (I - S^{-1}T(S^{-1})^*)\omega$ **and** $\omega_2 = -(S^{-1})^*\omega$**,**

Thank you very much !

The 6th Jikji Workshop:
Infinite Dimensional Analysis and Quantum Probability

January 8–12, 2011

Chungbuk National University (Cheongju 361-763, Korea)

http://crs.chungbuk.ac.kr/ hhlee/Jikji2011.html

Arrival Date: 7 (Friday) January, 2011

Departure Date: 13 (Thursday) January, 2011

- **Winter School: 8 (Saturday) $\sim$ 9 (Sunday) January**

This winter school consists of special lectures covering recent developments of infinite dimensional analysis and quantum probability, with wide applications to various research fields, and discussions for future directions.

Lecturers

- **L. Accardi (Centro Vito Volterra)**
- **K. B. Sinha (JNCASR)**

- **Workshop: 10 (Monday) $\sim$ 12 (Wednesday) January**

• Organizing Committee:

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– Hun Hee Lee (Chungbuk National University): hhlee@chungbuk.ac.kr

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• Registrations: For convenience of organizing the workshop, all participants are kindly asked to submit the registration form until October 31, 2010 to a member of organizers by e-mail. There is no registration fee. Kindly note that, from the limit of our budget, the total number of participants is restricted, and so if you are interested in participating in the 6th Jikji Workshop, then please send an e-mail to any organizer in advance.

• Accommodations: The accommodations for all participants will be provided by the organizers. Unfortunately, we can not support for the travels of all participants due to the limited grants, however we might be able to support travel expenses for a limited number of participants. In case you need support for the travel expenses, then please contact the organizers in advance.